$$C = \xi_1\xi_2\xi_3 + \Sigma\, \xi_i\eta_i^2 + k\Sigma x_1\xi_1\eta_1 + k\Sigma x_1\eta_2\eta_3 + k^2\Sigma x_1x_2\eta_3 + k^3x_1x_2x_3,$$

in which $n = v + kx_1x_2x_3$, while the quotients

$$\xi_i = \tfrac{1}{2}\frac{\partial^2 v}{\partial x_i^2},\ \eta_1 = \tfrac{1}{2}\frac{\partial^2 v}{\partial x_2\,\partial x_3},\ \cdot\ \cdot\ ,\ \eta_3 = \tfrac{1}{2}\frac{\partial^2 v}{\partial x_1\,\partial x_2}$$

have integral coefficients. The points of inflexion of $n = 0$ are its intersections with $C = 0$. Although C is not a covariant, it forms with n a covariant pencil, since C is transformed into a linear function of n and C.

[1] Hurwitz, *Arch. Math., Leipzig*, ser. 3, 5, 25, (1903).

[2] Dickson, *Madison Colloquium Lectures*, American Mathematical Society, (1914).

[3] Dickson, *Trans. Amer. Math. Soc.*, 15, 497, (1914).

[4] An advance in the theory of seminvariant leaders of covariants of quadratic forms has been made recently by the writer, *Bull. Amer. Math. Soc.*, January, 1915.

[5] Dickson, *Trans. Amer. Math. Soc.*, April, 1915.

[6] MS. offered Aug. 4, 1914 to *Amer. J. Math.* To appear April, 1915.

THE SYNTHESIS OF TRIAD SYSTEMS Δ_t in t ELEMENTS, IN PARTICULAR FOR $t = 31$

By Henry S. White

DEPARTMENT OF MATHEMATICS, VASSAR COLLEGE

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Purely theoretical interest first led to the study of triad systems Δ_t in t elements; systems of threes or triads, that is, in which every possible pair of elements is found in some one triad, but only one. Their relation to other objects of research in algebra and geometry began to appear when Noether (1879) pointed out the peculiar nature of a resolvent equation of the seventh degree which had been found by Betti, Hermite, and Kronecker in discussing the transformation of the seventh order of elliptic functions. This modular equation has roots related in triads like the Δ_t. Hesse had shown earlier, in plane curves of the third order, that the nine inflexion points lie by threes on twelve lines, thus exemplifying a Δ_9. Noether succeeded also in connecting the Δ_7 with the important sets of double tangents to the plane quartic curve which Aronhold had introduced under the name of *Siebenersysteme*. With those important applications in hand, and ten or twelve known Δ_{15}'s to serve as further data, mathematicians took up with renewed interest the question whether there are actual triad systems for every suitable member t of elements, i.e., for $t = 13, 15, 19, 21, 25, 27$, etc.; or precisely, $t = 6k + 1$ or $6k + 3$.

Although Reiss in 1859 had answered this question in the affirmative, constructing one system for every value of t, his work was overlooked until the same question was settled independently in 1893 by E. H. Moore. Moore's methods, based on more penetrating analysis than Reiss's, established at least two distinct sorts of systems for every t above 13, and led him to forecast definitely that the number of such systems would be found a rapidly increasing function of the number of elements, t. As to the sole doubtful number, $t = 13$, Zulauf, a pupil of Netto, found that there are two different systems, and others soon proved that there are no more than two.

After the lapse of ten years or more, Miss L. D. Cummings has now shown conclusively (1914) the existence of at least 24 distinct triad systems in 15 elements. These 24 include all that had been found before and as many more new systems, with their differences now for the first time rigorously demonstrated. All but one (viz., Heffter's) of these 24 exhibit what I call odd-and-even structure. The odd part are the elements appearing in seven triads that constitute an included triad system Δ_7, which may be termed the head in its Δ_{15}. Heffter's Δ_{15} is at present the only headless system in 15 elements whose description has been published.

Since the appearance of Miss Cummings' dissertation, I have applied a new method for constructing all possible Δ_{15}'s which can be transformed into themselves by any substitution among their elements,—all whose group is above the identity. By this means I find a considerable additional number of systems, all headless. These new Δ_{15}'s I now employ in attacking the question, how many distinct systems Δ_{31} are there in 31 elements? If there were but few, then it would be desirable to compile a complete census of them as further substratum for a general theory. But what I find is that even the restricted class selected for this study are far too numerous for detailed exhibition, their number being greater than 10^{13}. This result is attained through a new theorem, whose generality is significant of further possibilities.

The theorem, specialized for application, is this. *If among the triads of a system* Δ_{31} *there occur two complete systems* Δ_{15} *and* Δ'_{15}, *then there is a* Δ_7 *whose seven triads, and no other triads or elements, are common to* Δ_{15} *and* Δ'_{15}. Conversely, *if a* Δ_{31} *contains a headless* Δ_{15}, *it can contain no other triad system* Δ'_{15}; nor indeed any other larger than a Δ_7, and even such a Δ_7 will have one triad from the Δ_{15}.

Odd-and-even structure in a Δ_{31} consists in this: its 155 triads include 35 that form a Δ_{15} in 15 elements, and of the remaining 16 elements two are found in each of the other 120 triads, along with one element from

the 15. Thus every triad has an odd number, 3 or 1, of the *odd set* of 15, and an even number, 0 or 2, of the *even set* of 16. Any one of the odd set is found therefore in triads with 8 pairs from the even set, and these pairs can be arranged in 15 columns, an 8 by 15 array. Every odd element is found also with 7 pairs from the odd set. This leads to the tabulation of 15 columns of 7 pairs each, ranged above the columns of the 8 by 15 array. Every column is marked then by one odd element above it; the upper partial columns exhibit the head, or Δ_{15}.

Head and array form a convenient mode for constructing Δ_{31}'s that are to have odd-and-even structure. If the head, the Δ_{15}, is itself headless, this tabulation is unique for that Δ_{31}. *I study here exclusively these odd-and-even Δ_{31}'s whose head is a headless Δ_{15}.* Given any one such Δ_{31}, tabulated, many others can be obtained by shifting the columns of its 8 x 15 array while the head is kept stationary. To apply this method and to count the distinct Δ_{31}'s that will be produced, one must know the groups $G_{d'}$ and $G_{d'}$, belonging to the head and to array respectively.

The number of resulting Δ_{31}'s is certainly not less than 15! divided by the product, d d′, of the orders of the groups belonging to the head and to the array respectively. These orders are small, whence the resulting Δ_{31}'s are very many.

Incidentally, if d and d' are relative primes, the resulting Δ_{31}'s must be of the peculiar kind having no automorphic substitutions; i.e., their group is reduced to the identity. Such cases occur, e.g., with $d = 2$ and $d' = 3$. Full details are to appear in the *Transactions of the American Mathematical Society* for January, 1915.

THE Φ-SUBGROUP OF A GROUP OF FINITE ORDER

By G. A. Miller

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

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A set of λ operators $s_1,s_2,..,s_\lambda$ of a finite group G is called a set of generators of G provided there is no subgroup in G which includes each of these operators. When these operators satisfy the additional condition that G can be generated by no $\lambda - 1$ of them the set is said to be a *set of independent generators of G.* Those operators of G which can appear in none of its possible sets of independent generators constitute a characteristic subgroup, which was called by G. Frattini *the ϕ-subgroup of G.* See *Rend. Acc. Lincei,* ser. 4, 1, 281 (1885).